Losing an i: on the Naturalness of Complex Numbers for Quantum Mechanics[*]

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Abstract

It is often said by physicists that whereas in classical physics complex numbers are a useful but dispensable mathematical tool, in quantum mechanics complex numbers become necessary or essential to formulating the theory. It is not often said why, or in what way, complex numbers are necessary here. Even among those who proffer a reason, there is nothing like a consensus.

I argue that an illuminating reason for the peculiar well-suitedness of complex numbers flows from distinctively quantum phenomena involving spin. This is an elucidating reason for the special role played by complex numbers in quantum mechanics vis-à-vis classical physics, flowing as it does from some simple yet distinctively quantum mechanical phenomena, yielding insight into the nature of a quantum mechanical world and the mathematical structure of a physical theory that governs it.

1. Introduction

It is common to hear physicists say that complex numbers are essential to quantum mechanics as opposed to classical physics, where they are a convenient but dispensable mathematical tool. Nobelist Chen Ning Yang remarks that in classical physics, complex numbers were used as a "computational aid"; but in quantum mechanics, "the situation dramatically changed. Complex numbers became a conceptual element of the very foundation of physics" (1987, 54). Freeman Dyson says the imaginary unit in Schrödinger's equation shows that "nature works with

[*]Thanks to Laura Ruetsche for the pun.

complex numbers and not with real numbers" (2009, 213). J. J. Sakurai, in his textbook, calls the introduction of complex numbers "an essential feature in quantum mechanics" (1994, 27). Schrödinger himself found the appearance of complex numbers "unpleasant" and "directly to be objected to" (Przibram, 2015, 62), and tried to reformulate things without them; but he eventually abandoned that attempt (for reasons that aren't completely clear), concluding that, "the wave function has to be regarded as essentially complex" (Schrödinger, 2020, 216, n. 6).[1]

It is much less common to hear physicists say why, or in what way, complex numbers are necessary in quantum mechanics as opposed to classical physics. And among physicists who do say something about this, there is nothing like a consensus—even though just about every big name physicist associated with the theory since its inception has puzzled over this, including the likes of Schrödinger, Ehrenfest, Pauli, Bohm.[2] Instead, this is treated as an outstanding question.[3]

Despite the frequency with which it is said, though, it cannot be that complex numbers are truly necessary or essential, genuinely indispensable. Any physical theory can be formulated in different, mathematically equivalent ways. (A theory can even be formulated in ways that are mathematically inequivalent in some sense, given the appropriate interpretational stipulations.) It is also a straightforward mathematical fact that any expression involving complex numbers can be restated in terms of real ones. Just think of expressing a complex number in terms of its real and imaginary parts, each themselves real. Complex numbers can generally be regarded as ordered pairs of real numbers obeying certain algebraic relations. It seems it should be possible to restate any formalism using complex numbers equivalently in terms of real ones. (Schrödinger himself hit upon a real version of his equation.[4])

Indeed, there are real formulations of quantum mechanics available

[1]Discussion in Chen (1989); Karam (2020a); Callender (2023).

[2]Karam (2020b) surveys four different ideas that can be found in physics textbooks.

[3]This is expressed from Schrödinger to Wigner (1960) through today (Halvorson, 2003; Aleksandrova et al., 2013; Maynard et al., 2015; Karam, 2020a,b). It is evident in the many projects aimed at reformulating the theory without complex numbers (n. 7).

[4]See Callender (2023) for advocation of this equation for solving a puzzle about time reversal in quantum mechanics.

(and formulations in terms of quaternions).[5] These show by construction that it cannot be strictly necessary to use complex numbers.[6] (In recent decades there have been various "reconstruction projects" aiming to rebuild the theory from the ground up, on the basis of new fundamental principles, usually with a different mathematical framework, in a way that aims to explain the use of complex numbers in the standard theory.[7] My focus is their role within the standard mathematical framework.)

At the same time, it doesn't seem right to completely discount the significance of complex numbers, either. Physicists have been putting their finger on something characteristic about the theory when they suggest that complex numbers are playing a newly distinctive role. Complex numbers do seem particularly natural or well-suited to quantum mechanics, in a way that suggests there is something about the nature of quantum mechanical systems that underlies their naturalness; something distinctively quantum mechanical that such a formulation is latching onto. (Note, too, that formulations that do away with complex numbers, such Stueckelberg's (1960) (n. 5), generally possess structure that effectively encodes the algebra of complex numbers in terms of real ones (it is even called a "complex structure"). So although complex numbers might be jettisoned by these alternatives in some sense, there is another sense in which their basic nature or structure has not been entirely eliminated. We might say that either complex numbers or a complex structure are natural for the theory, complex numbers themselves particularly so.[8])

In all: it cannot be that complex numbers are necessary or essential, despite frequent assertions that they are. However, they do yield a for-

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[5] Stueckelberg (1960); Ashtekar and Schilling (1999); Hestenes (2003); McKague et al. (2009). Adler (1995) uses quaternions. There are mathematical arguments suggesting that quantum theory can (only) be formulated in real, complex, or quaternionic Hilbert spaces (Stueckelberg, 1960; Solèr, 1995; Myrheim, 1999).

[6] Compare Norsen (2017, 116); Callender (2023, 842).

[7] A resurgence of interest in such projects began with Hardy (2001). A different kind of reconstruction project is in Barandes (2023).

[8] Earman (2024) suggests that something physically significant is lost in any real formulation, on a certain understanding of the relationship between a real quantum theory and its "complexified" version. He concludes that a real version won't always be empirically equivalent to ordinary complex quantum mechanics (and that Stueckelberg's theory in particular is empirically inadequate).

mulation that is particularly natural or perspicuous, in a way that feels very different from their role in classical physics.

This leads to a version of the question physicists have been asking: what is it about quantum mechanics and the kind of world it describes, as opposed to classical physics and the kind of world it describes, that makes complex numbers so well-suited or natural? Why, in the case of quantum mechanics, are complex numbers "so admirably appropriate to the objects of reality" (in a phrase from Einstein (1922))?

This is the question I address here. I take it that an answer should locate something about the physical nature of a quantum mechanical world or system, as opposed to any classical world or system, that makes complex numbers so natural for characterizing its physics.

The answer I outline will point to phenomena involving spin. This is not unheard of; something like this idea can be found in scattershot ways in a few places, albeit with highly varying approaches.[9] Yet it is rarely mentioned, including in the context of inquiring into the special role of complex numbers. This is surprising, since from among the morass of ideas out there, it is notably elucidating, because mathematically, conceptually, and physically clear. It is a reasonably simple idea that nonetheless directs us toward general features of the mathematical structure needed for a theory to encompass distinctively quantum phenomena in a perspicuous way. That is what I aim to show here. (My aim is to advertise the idea as an answer to this outstanding puzzle; future work will explore it in further detail.)

A few notes. For simplicity, I restrict attention to nonrelativistic particle quantum mechanics, and will not assume any particular theory of quantum mechanics: I remain neutral on that. (The nature of spin varies for different theories, but the basic idea vis-à-vis complex numbers won't.) I am also going to assume some version of scientific realism, some view that is happy to inquire into the nature of things that a successful mathematically formulated physical theory is latching onto, though certain anti-realists can buy into this sort of inquiry as well.

Additionally, I am going to set aside something else often said: that we can only ever measure real-valued quantities, or that genuine physical

[9] Hestenes (1975, 2003) (cf. Baylis et al. (1992)); Sakurai (1994); Bowman (2008); Sivakumar (2012); Townsend (2012).

quantities can only be represented by real numbers. On this idea, complex numbers aren't the kinds of things that can be probed for what they tell us about physical reality; they can only ever be indirect calculational tools. I assume that any type mathematical object can be used to represent genuine aspects of physical reality, complex numbers being no worse off.

I won't here address every rationale for the centrality of complex numbers that can be found. Rationales are not very often given, but those that are vary widely in outlook and approach (unsurprisingly, given the wide range of approaches to quantum mechanics itself). Instead, I will discuss one idea that has some initial plausibility, and use it to motivate an alternative that, I argue, is more elucidating and explanatory. Note too that I won't be getting at the question of complex numbers in a way that will be familiar to philosophers: by aiming to nominalize the theory. That approach tries to show that we don't need numbers or abstract objects at all to formulate a theory (in order to show that don't have to ontologically commit to them), which would then explain the usefulness of the mathematical objects we familiarly employ.[10] My aim is just a bit different. I am not concerned with nominalism, but with exploring why complex numbers are distinctively well-suited to describing quantum as opposed to classical phenomena. If you like, my aim is more physical than metaphysical.

As a final initial note, I want to reiterate that the question of complex numbers has been puzzled over on and off since the early days of quantum theory, with very different sorts of answers being given, and no one clear or agreed-upon idea having emerged. I aim to show that there is an elucidating idea that has, as it were, been hiding in plain sight.

2. One answer: dynamic complexity

Let me mention one idea that is sometimes said, which will be instructive.

But first, let me set aside one more thing. You might be wondering about my claim that complex numbers can't be necessary for quantum mechanics. It has recently been proclaimed that not only are they necessary, but this has been experimentally verified. Two different groups of

[10] See Balaguer (1996); Chen (2018).

physicists claim to have experimentally confirmed predictions for certain phenomena involving multi-particle systems, which they say cannot be reproduced by any real quantum theory.[11] This has gotten a lot of press, in articles with titles like: “Complex Numbers are Essential in Quantum Theory, Experiments Reveal” (Padavic-Callaghan, 2022).

Given the mathematical facts about complex numbers in particular, and the nature of mathematical representation in general, it is hard to see how this can be correct. And in fact, those researchers make certain assumptions which show the conclusion to be weaker than having demonstrated the true impossibility of having an empirically adequate quantum theory without complex numbers. For instance, they assume the usual tensor-product rule for representing states of combined systems, which Stueckelberg’s formulation rejects. It’s also not clear that their argument can be extended to all solutions to the measurement problem, like Bohm’s theory.[12] At most, they have shown that one way of reformulating the standard theory using only real numbers won’t work; not that *no* real version will fail.[13] I am going to set that aside here.

One rationale for the necessity of complex numbers sometimes given is that the time-dependent Schrödinger equation explicitly contains the imaginary unit, so that solutions must be complex-valued functions. In classical physics, by contrast, although we *can* always use complex numbers for various purposes, we never *have* to, since the dynamical equations don’t explicitly contain i.

Bohm (1951) says this, noting that in classical physics, the real and imaginary parts of any complex function we might use always remain independent, described by equations that are decoupled. This allows us to take only the real part to be physically significant, so that complex numbers are just “an auxiliary device.” But (as Bohm shows) the Schrödinger equation cannot be rewritten as two uncoupled equations that independently govern the real and imaginary parts of a complex

[11] Renou et al. (2021a); Chen et al. (2022); Li et al. (2022).

[12] Renou et al. acknowledge as much, but only in a single paragraph in their *Nature* article and in supplementary material that is easy to miss (Renou et al., 2021b). A more recent article continues to assert that the relevant experimental results “cannot be explained through real quantum models” (Renou et al., 2023, 64).

[13] Earman (2024) (n. 8) is more sympathetic.

wavefunction. Hence both the real and imaginary functions must be physically significant. This is to point to what Julian Barbour (1993) calls the theory's "dynamic complexity."

This first of all cannot account for the true necessity of complex numbers. This is for reasons mentioned (and Bohm essentially acknowledges). But it is worth explicitly noting that this holds of the Schrödinger equation, which can be restated in real terms; and not only as the two coupled real equations that Bohm discusses, but also as a single higher-order real equation, in terms of one real variable that removes the coupling (the one Schrödinger discovered).

More than that, the idea is not sufficiently illuminating. Pointing out *that* the standard version of the time-dependent Schrödinger equation explicitly contains i, so that solutions will be complex-valued functions, doesn't help explain *why* this is—what it is about the physical nature of quantum systems that underlies the appearance of the imaginary unit. We are left with the feeling that stating the equation in terms of complex numbers is more of a convenient shorthand, as it is in classical physics.

Still, the coupling of the two real equations that are equivalent to the standard Schrödinger equation is a hint that quantum mechanics requires the extra information that's contained in a complex function or equation, with both real and imaginary parts, as opposed to just a real one. It is this idea of some "extra information" about quantum mechanical systems that needs to be captured by the formalism that is instructive.

3. A better answer: kinematic complexity

A different idea is more illuminating, flowing from the nature of spin. The basic idea is that there must be enough "room" in the formalism to represent all the different possible spin states of a system and the relations among them, and a formulation in terms of complex numbers does this in a particularly natural and direct way. There is "extra information" in the facts about spin states that is well-represented in terms of complex numbers, or a complex Hilbert space. This points to the theory's "kinematic complexity" (also in Barbour's (1993) phrase) to the statespace structure needed to represent different possible states in a perspicuous way.

The idea is not hard to see, and is something that anyone with an introductory-level quantum course will be familiar with. But the simplicity is deceptive, and belies real insight to be had.

Think of some simple Stern-Gerlach-type experiments. (I am going to describe things to be as neutral as possible among different theories of quantum mechanics.) Suppose we are going to investigate the behavior of electrons. (Using electrons requires certain idealizations that won't matter here.) We will send electrons that have not been prepared in any special way through an inhomogeneous magnetic field and toward a fluorescent screen. Spots on the screen will light up where an electron lands. Repeat the experiment many times with the apparatus oriented in different directions in space, particularly different orthogonal directions we will call X, Y, and Z.

The first thing to notice is that we only ever see two different kinds of outcome. However the Stern Gerlach magnet is oriented in space, electrons will emerge moving in one of two directions, giving rise to spots on the screen at two different locations. Each electron is deflected to land at a location upward on the screen (up in the direction along which the inhomogeneous magnetic field points), or downward, with just about half landing at each. It also seems random which electron goes which way, so that each individual electron has a 50-50 chance of being deflected up or down, no matter which direction we orient the apparatus.

Since no classical sort of "intrinsic angular momentum" should behave this way—classically, we expect to see a continuous range of deflections—we are led to posit a distinctively quantum mechanical property, the spin. Spin can be measured in any spatial direction, it has a component in any direction, and for electrons, the spin in a given direction can take one of two values, "up" or "down" (with equal probability for each).

The second thing to notice is what happens when we send electrons through a sequence of two or three Stern-Gerlach magnets in a row, which may be oriented in different spatial directions; particularly what happens when the first magnet is oriented in a given direction, the second in a direction orthogonal to the first, and the third oriented in the same way as the first. In other words, take the electrons that emerge from the first magnet within the upward beam and send them through a magnet that is oriented at a right angle to the first; similarly for the downward beam; and analogously for the beams from the second magnet.

Repeating this many times leads us to conclude, first, that electrons can have, or be in, a definite or determinate state of spin, in that we always find the same result, up or down, whenever the magnets are all oriented in the same way (an X-spin up electron continues to come out up); and second, that the value of the spin along one direction is uncorrelated with the value along an orthogonal direction. Whenever an electron is found to have a certain value, up or down, of spin in some direction, there is a 50-50 chance the spin will have the same value if we send the electron through a magnet oriented at a right angle immediately afterward. (If it is initially Z-spin up, there is a 50-50 chance it will come out X-spin up.) The upward- and downward-directed beams coming from the first magnet each split into two equal halves when passing through the second one; and again, similarly, through the third.

Taken all together, these experiments suggest that a determinate state of spin along one direction is an indeterminate state of spin along an orthogonal direction, in yielding a 50-50 probabilistic prediction for the spin to be up or down in that direction; and this is the case even if we had earlier, with the first magnet, found a particular value in that direction. Even if an electron is found first with X-spin up, by the time it reaches the third magnet it is only 50-50 to be X-spin up.

Now think of how we represent what we have found in these experiments mathematically, being particularly careful to pay attention to what each mathematical feature of the representation corresponds to physically. Using complex numbers, the following things will hold of the statespace representation. (Remember that I am assuming the standard mathematical framework, so that this will be a vector space, in particular a Hilbert space. I am not here addressing the question as to why we use this mathematical framework altogether.)

First, the statespace, the spin space, is two-dimensional. This mathematical feature corresponds to the physical fact that there are two different possible outcomes or states or values for any Stern-Gerlach experiment with electrons, oriented in any spatial direction: spin-up and spin-down.

Second, the two states of determinate spin in a given direction—the eigenstates of spin, "up" and "down"—are represented by two orthogonal (really orthonormal), vectors in the statespace. Each pair of vectors corresponding to a determinate state of spin-up and spin-down in a given direction—each pair of eigenvectors of the corresponding operator—forms a distinct orthonormal basis in the statespace. This mathematical feature corresponds to the physical fact that an electron is always found to have spin up or down in any direction, and that these are distinct, mutually exclusive physical states or outcomes (given the Born rule and the usual probabilistic interpretation of the inner product)—that whenever we find X-spin up, say, there is no chance of finding X-spin down if we repeat the experiment in that direction. It's also what allows us to represent all the possible spin states in any direction: each spin component corresponds to a distinct orthonormal eigenbasis. This is a genuine *statespace* for spin.

Third, a determinate state of spin in one spatial direction is represented by an equally-weighted sum of the eigenvectors of spin in either orthogonal direction. The eigenbasis corresponding to spin along each of the three spatial directions are at 45° angles with respect to one another in the statespace. This corresponds to the physical fact that a state of determinate spin in one direction is equally likely to yield spin-up or spin-down along either orthogonal direction, along with the fact that ordinary space is three-dimensional.

Fourth, the vector representing a state of determinate spin in a given direction is a *different* equally-weighted sum of eigenvectors of spin in either orthogonal direction, from the equally-weighted sum of eigenvectors of spin for each orthogonal direction; the X-spin up eigenvector must be a different sum of Y-spin eigenvectors from that of Z-spin up, say. This corresponds to the physical fact that a determinate state of spin in one direction is a distinct physical state from that of spin in either of the two orthogonal spatial directions, with different physical properties, exhibiting different behavior, for instance if we send electrons through a Stern Gerlach device oriented in that direction. Different states are represented by distinct vectors, which take on different expressions when expanded in terms of a given basis.

Thus, each spin component, in each of three ordinary spatial directions, corresponds to a distinct orthonormal eigenbasis in the statespace, each at a 45° angle with respect to one another. This can't happen in a two-dimensional real vector space. Complex numbers allow us to "fit" an additional pair of orthogonal vectors, at a 45° angle from two other pairs of orthogonal vectors, which are at a 45° angle from each other, into the two-dimensional statespace. (I have been emphasizing this geometrically, but one can show it algebraically. Formulate the matrix representation for the three orthogonal spin operators and corresponding eigenvectors in a given basis: you cannot do this using only real coefficients.[14])

There is in this sense "extra information" concerning the physical facts about spin states and their relationships to one another that can't be straightforwardly squeezed into a real vector space, but is well-represented using complex numbers.

That said, a complex Hilbert space isn't strictly required. Just as a single complex number can be captured by means of two pieces of information in an ordered pair of real numbers, the structure of a two-dimensional complex vector space can be captured by a four-dimensional real vector space; which suggests that it should always be possible to represent things in terms of either a real or a complex Hilbert space, with the right structure. You just need something that mimics or encodes the structure of the two-dimensional complex vector space.

Still, the doubling of the number of dimensions of the statespace (and the operators representing observables) in the real formulation results in having to impose additional constraints in order to delimit the allowable states or transitions.[15] The formulation in terms of complex numbers is in this way more perspicuous or direct. It doesn't have a type of excess mathematics in the representation that will be dispensed with by hand in the corresponding real formulation.

4. Conclusion

This is a notably physically clear reason for the special role played by complex numbers, flowing from reasonably simple yet "inherently quantum mechanical" phenomena (McIntyre, 2012, 28); it is for that reason especially elucidating. It points to something about the physical nature

[14] See Townsend (2012); Karam (2020b). The above train of thought is in a similar vein to McIntyre (2012, ch. 1); Sivakumar (2012) but with a different bent and upshot.

[15] Aleksandrova et al. (2013, 2); Vedral (2023). The many-interacting-worlds approach, perhaps most naturally formulated using real numbers, likewise requires a further constraint on the space of states: Sebens (2015, sec. 6).

of quantum mechanical systems that underlies the distinctive naturalness of complex numbers for describing them.

This is a more general idea than initially meets the eye. It is true that complex numbers are used to characterize state vectors or wave-functions without spin, and for particles other than electrons. However, all fundamental particles (aside from the Higgs boson) have spin. And the basic idea of being able to "fit" mathematical representations of all the possible states into the appropriate statespace should apply to other quantum mechanical quantities, when we consider incompatible ones.[16] (Nor is it off the table to think the appearance of i generally flows from spin, as suggested in the work of David Hestenes, who uses a different mathematical foundation that is outside the scope of this paper.)

For that matter, it is sometimes said that systems with spin (electrons especially) are the simplest truly quantum systems.[17] If the simplest paradigmatically quantum system underscores the use and role of complex numbers, then this is as good an explanation as any. (Compare Pauli's own introduction of spin, a "peculiar non-classically describable two-valuedness," by doubling the number of available electron states due to a "hidden rotation" (quoted in Pais (1991, 201)).) Indeed, we can also turn things around to see this as elucidating the way in which spin is a distinctively quantum mechanical quantity, as no classical quantity generates this type of trouble for the mathematical representation. Compare a remark from Shelly Goldstein (2021), which suggests this:

> Spin is the canonical quantum observable that has no classical counterpart, reputedly impossible to grasp in a nonquantum way. The difficulty ... is that there is no ordinary (nonquantum) quantity which, like the spin observable, is a 3-vector and which also is such that its components in all possible directions belong to the same discrete set. The problem, in other words, is that the usual vector relationships among the various components of the spin vector are incompatible with the quantization conditions on the values of these components.

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[16]On this, see Halvorson (2003). The above is in the spirit of Halvorson's note, while differing in emphasizing the experimental results underpinning the conclusion, and in allowing that a real vector space can do the job, just not as well.

[17]McIntyre (2012, 28); Susskind and Friedman (2014, 3–4); Bricmont (2016, 21).

The mathematics needed to perspicuously represent spin facts underscores the distinctiveness of spin. Indeed, the explanation from kinematic complexity, from the structure of the statespace, arguably underlies the theory's dynamical complexity (something I must explore elsewhere).

This might still strike you as all *too* easy; or as something that physicists, who bandy about the Bloch sphere, a geometric representation of the relationships among spin states and directions in ordinary space, are aware of. But the simplicity is deceptive. It just goes to show how elucidating the idea is, pointing to core aspects of the theory. As well, it has not permeated the literature directed at the question of the role of complex numbers in quantum mechanics, where all manner of explanations continue to be offered. So if in the end I have simply reminded us of something that has been hiding in plain sight, that is progress.

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